Particle Accelerators
1980, Vol. 10, pp. 73-81
0031-2460/80/1001-0073$04.50/0

© Gordon and Breach, Science Publishers, Inc.
Printed in U.S.A.

COHERENT INTERACTION BETWEEN WAVES AND PARTICLE STREAMS

J. D. LAWSON

Rutherford Laboratory, Chilton, Oxon, England

(Received April 26, 1979)

The interaction between beams of particles and traveling electromagnetic waves forms the basis of a large number of practical devices. Well-known examples are particle accelerators, microwave tubes, the cyclotron maser, and more recently, the free-electron laser. Such interaction is also responsible for the phenomenon of Landau damping which occurs in plasmas and intense accelerator beams. In this paper the basic characteristics of the wave-particle interaction are examined in a general way and illustrated with reference to the above examples.

I. INTRODUCTION

Many devices depend for their operation on the coherent interaction between traveling electromagnetic waves and streams of particles. Such an interaction forms the basis of linear particle accelerators, traveling-wave tube amplifiers and the free-electron laser. The spatial and temporal variation of such waves is characterized by the factor $\cos(\omega t - kz + \psi)$. In some devices, for example cyclic accelerators, magnetrons and the cyclotron maser, the wave has angular rather than linear velocity so that kz in the argument of the cosine is replaced by $n\theta$. Coherent interaction between waves in a plasma or intense accelerator beam and some of the particles that constitute it gives rise to the phenomenon of Landau damping.

Detailed theories describing these devices tend to be complicated and specialized. The object of the present paper is to present a simple, general treatment of the interaction of particles with traveling waves in terms of a few characteristic parameters. These general parameters may then be expressed in particular form for particular applications. The objective is not to enable calculations of immediate practical value to be made, but rather to illustrate common features not always apparent in the conventional theories.

Three regimes of operation may be distinguished; in the first of these, the particle density is sufficiently dilute that the amplitude of the external wave is not affected. In the second, the presence of the particles affects the wave amplitude, but direct interaction between particles can be neglected. In the third regime, interaction between the particles is significant; when this is so the ratio of frequencies associated with the collective effects to other characteristic frequencies of the system is not small. The analysis in this paper will be confined to the first regime. Extension to the second is relatively straightforward; the third, however, requires a more sophisticated approach.

II. SINGLE PARTICLE IN A UNIFORM WAVE

The simplest situation, to be considered first, is that of a particle and wave both traveling in the z direction, where the wave has an electric field of the form

$$E_z = E_0 \sin(\omega t - k_z z) \tag{1}$$

and the particle velocity is $\dot{z}$. Relativistic units will be used, so that $\dot{z}$ is written $\beta_z c$, and $\gamma_z = (1-\beta_z^2)^{-1/2}$. If ϕ represents the phase of the particle with respect to the zero of the wave, (Fig. 1) then

$$\phi = \omega t - k_z z \tag{2}$$

$$\dot{\phi} = \omega - k_z \beta_z c. \tag{3}$$

The particle obeys the equation of motion

$$eE_z = \frac{d}{dt}(\beta_z \gamma_z m_0) = \dot{\beta}_z m^* c, \tag{4}$$

where m^* is defined as the effective mass, to be discussed later. Here it is equal to $\gamma^3 m_0$, the long-

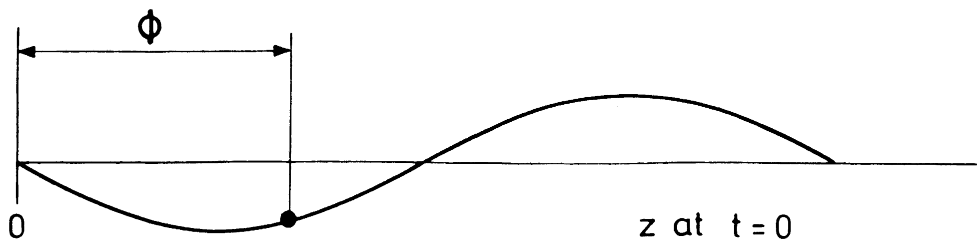

FIGURE 1 Diagram to illustrate the definition of ϕ.

itudinal relativistic mass. [It is readily verified that $d(\beta\gamma) = \gamma^3 d\beta$]. From Eqs. (1)–(4),

$$\frac{E_z}{E_0}\frac{\ddot{\phi}}{\dot{\beta}_z c k_z} = -\sin\phi \tag{5}$$

This may be written

$$\ddot{\phi} + \Omega^2 \sin\phi = 0, \tag{6}$$

where

$$\Omega^2 = k_z e E_0 / m^* \tag{7}$$

Integrating Eq. (6), we have

$$\dot{\phi}^2 - \dot{\phi}_0^2 = 2\Omega^2(\cos\phi - \cos\phi_0), \tag{8}$$

where ϕ_0 and $\dot{\phi}_0$ are the values of ϕ and $\dot{\phi}$ at $t = 0$. A plot of $\dot{\phi}$ versus ϕ is shown in Fig. 2. It is well known that this equation represents the motion of a rigid pendulum. Closed orbits in the phase plane represent oscillatory motion, with frequency varying from Ω for small-amplitude motion to zero for a phase swing of 2π; open orbits on the other hand represent continuous rotation.

For the problem of wave and particle, a more direct mechanical analogy is provided by a ball rolling on a sinusoidally modulated surface moving with uniform velocity $\beta_z c$. Particles can oscillate in the troughs, or roll continuously forwards or backwards.

If $\dot{\phi} \ll \omega$, so that the two terms on the right-hand side of Eq. (3) are nearly equal, then the energy difference between the "synchronous" particle, for which $\dot{\phi} = 0$, and any other particle, is proportional to $\dot{\phi}$. This may be seen by noting that $\dot{\phi}$ is proportional to the velocity difference $\Delta\beta c$ between the two particles so that the energy difference becomes

$$\Delta\gamma m_0 c^2 = \Delta\beta(d\gamma/d\beta) m_0 c^2 = \beta\gamma^3 \Delta\beta m_0 c^2. \tag{9}$$

III. ENERGY INTERCHANGE BETWEEN THE WAVE AND AN ENSEMBLE OF PARTICLES

As a first step towards calculating the energy interchange between a particle beam and a wave, we now calculate the average, for particles uniformly distributed in ϕ_0, of $\phi(t) - \phi_0$ when the change of wave amplitude can be neglected. Such particles may be represented in the phase-plane at $t = 0$ as a horizontal line. As t evolves, the line becomes curved as the various particles move along the trajectories shown in Fig. 2. Such curves are shown in Fig. 3 for various values of t and $\dot{\phi}_0$. It is not in general possible to find a closed analytical expression for $< \dot{\phi}(t) - \dot{\phi}_0 >$ (where angle brackets denote the average). The expression for $\phi(t)$ involves elliptic integrals, and the average over ϕ_0 is best found numerically. This has been done, for example, by Planner[1] as a function of $\dot{\phi}_0$ and t.

For small times, however, an analytical expression can be derived for $< \dot{\phi}(t) - \dot{\phi}_0 >$. This is done in the Appendix. The result is

$$\begin{aligned} < \dot{\phi}(t) - \dot{\phi}_0 > &= \frac{1}{2\pi}\int_0^{2\pi} (\dot{\phi}(t) - \dot{\phi}_0) d\phi_0 \\ &= \frac{\Omega^2}{\dot{\phi}_0^3}(\cos\dot{\phi}_0 t - 1 - \tfrac{1}{2}\dot{\phi}_0 t \sin\dot{\phi}_0 t) \\ &= \frac{\Omega^4 t^3}{4}\left\{\frac{d}{dx}\left(\frac{\sin x}{x}\right)^2\right\}, \end{aligned} \tag{10}$$

where $x = \tfrac{1}{2}\dot{\phi}_0 t$.

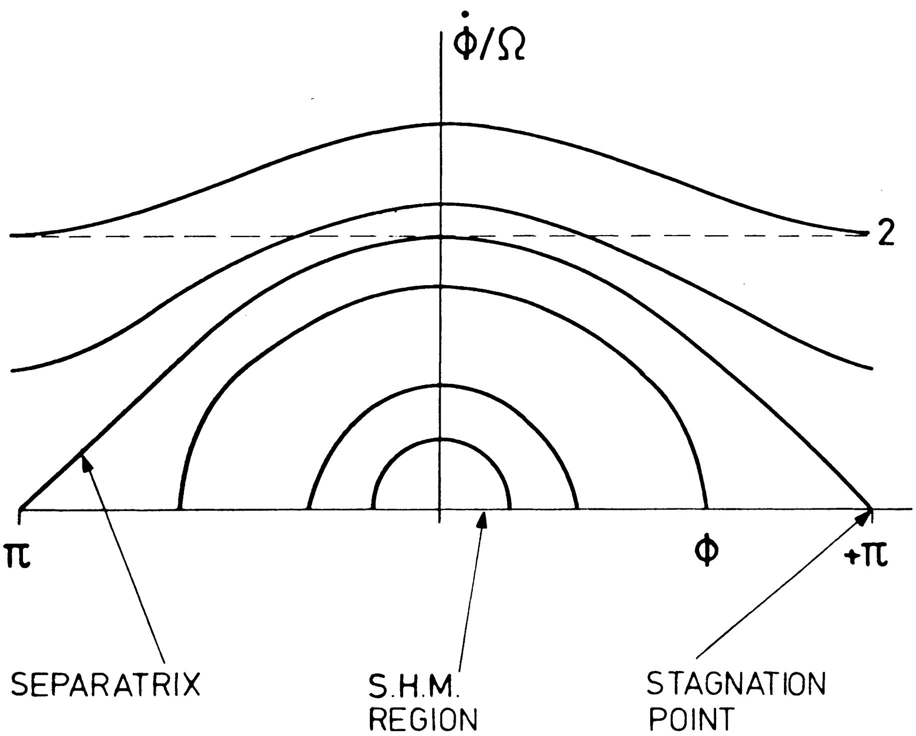

FIGURE 2 Phase-plane representation of the solution of Eq. 8. As time varies, particles move along the trajectories shown. Within the separatrix, the mean value of $\dot{\phi}$ for a given particle approaches zero when measured over a time $t \gg \Omega^{-1}$. Such particles are said to be trapped.

The range of validity of this expression is

$$t \ll 2\dot{\phi}_0/\Omega^2. \qquad (11)$$

At values of $\dot{\phi}_0/\Omega$ for which there are values of $\dot{\phi}_0$ within the separatrix, this represents a fraction of the oscillation period $2\pi/\Omega$; at large values of $\dot{\phi}_0/\Omega$, on the other hand, it represents many cycles of oscillation. For even shorter times, $t \ll 1/\dot{\phi}_0$, Eq. (10) simplifies to

$$< \dot{\phi}(t) - \dot{\phi}_0 > \simeq \Omega^4\dot{\phi}_0 t^4/24. \qquad (12)$$

Asymmetry between particles gaining and losing energy only appears in the fourth order in t.

IV. SOME PRACTICAL SYSTEMS. TRAVELING-WAVE TUBES AND PARTICLE ACCELERATORS

The wave–particle configuration which has been analyzed is characterized by rather few variables. There are two frequencies ω and Ω, describing the wave and small-amplitude phase oscillations respectively. The simplest device in this class is the nonrelativistic traveling-wave tube, in which space–charge forces in the beam are negligible. This has been analyzed for example by Webber.[2] His paper shows a pendulum and phase diagrams equivalent to Fig. 2. A useful description of a practical device requires inclusion of the growth of the wave along the tube; this requires an extension of the theory and the introduction of the concept of "series impedance," which relates the field intensity to the power flux. Numerical methods then become necessary; the results of such computations and comparison with experiment may be found in Ref. 2. In this device, $m^* = m_0$.

In the type of interaction described so far, the frequency and wave number of the wave (apart from the harmonic variation) are independent of position or time. In particle accelerators, on the other hand, either k_z varies with z or else ω varies with t, and the basic phase equation takes the form

$$\ddot{\phi} + \Omega^2(\sin\phi - \sin\phi_s) = 0, \qquad (13)$$

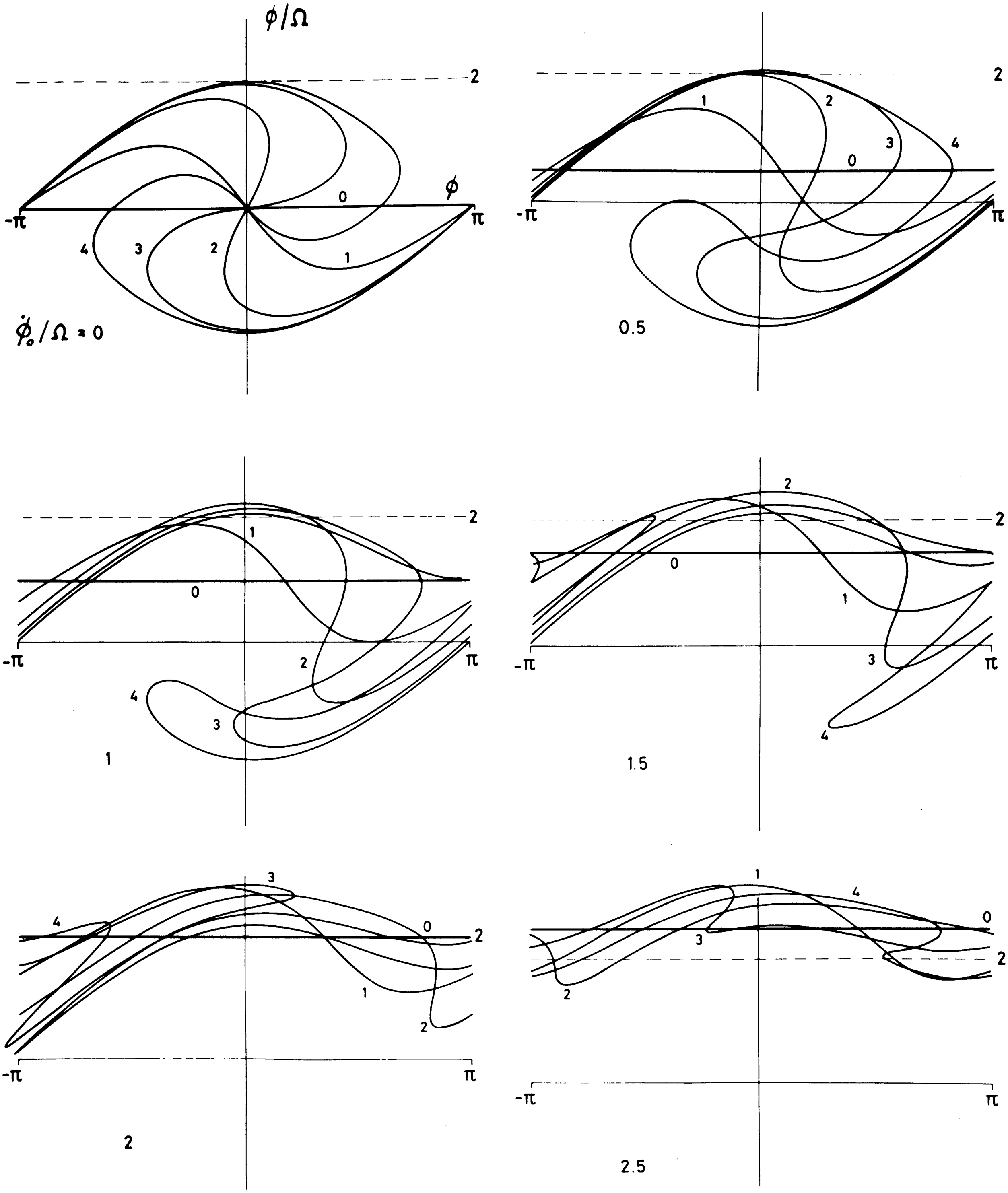

FIGURE 3 Evolution with time of an ensemble of particles which at time $t = 0$ are uniformly distributed in ϕ, but all have the same value of ϕ. Figures at the bottom left of each diagram denote ϕ/Ω at $T = 0$, and the distribution at subsequent times: the numbers on the curves represent time in units of Ω^{-1}.

where ϕ_s is the stable phase angle. This is now the equation of a biased pendulum, acted upon by a fixed applied torque such that the equilibrium position is at an angle ϕ_s to the vertical. Evidently the possible oscillation amplitude is reduced, and the separatrix in the phase plane does not extend from $-\pi$ to $+\pi$. Returning to the analogy of the ball moving on a modulated surface, this is now at an angle to the horizontal; the trapping range is less, but all the trapped particles on the average steadily increase in energy. Solutions of Eq. (13) are exhibited in texts on particle-accelerator theory.[3] In many accelerators the particles traverse a set of gaps; in an orbital machine the same gaps are passed many times. Such a field may be Fourier analyzed into a set of space harmonics with the same ω but different k_z; only that harmonic with phase velocity close to the particle velocity interacts coherently with it.

Sometimes, for example in the Stanford two-mile linac,[4] the field is so strong and variation of velocity with energy so small that γ varies by a very large factor during the period of one phase oscillation, whereas in the theory outlined here the fractional energy during the phase oscillation is small. Indeed, less than one oscillation occurs in that whole accelerator. It is interesting to note, however, that behavior described by the present theory does occur in linear accelerators, in the form of a backward-wave oscillator effect which can give rise to beam breakup.[5] A wave with transverse electric field, having wave length uncorrelated with the bunching of the electrons, is generated. This produces a sideways deflection of the electrons, which then strike the walls. This has been analyzed in detail by Wilson.[6] Equation (20) of Ref. 6 may be compared with our Eq. 10).

Two types of system have now been discussed. In the first, the wave system is uniform, $\phi_s = 0$, and the synchronous particle moves at uniform velocity; in the second, ϕ_s is finite, and the synchronous particle is continuously accelerated.

In general, for a particle accelerator

$$\Omega^2 = (heV \cos \phi_s / m^* C), \tag{14}$$

where C is the accelerator circumference, eV is the maximum volts per turn (equal to eCE_0 where E_0 is the electric field of Eq. (1)), and h is the harmonic number, equal to $C\omega/2\pi\beta c$, where ω is the frequency of the rf system. The effective mass is given by

$$m^* = \frac{\gamma m_0}{\dfrac{1}{\gamma^2} - \dfrac{1}{\gamma_t^2}} . \tag{15}$$

The quantity $\gamma_t m_0 c^2$ is the transition energy. Above this energy, which depends on the lattice structure of the accelerator, the mass is effectively negative. This means that if a particle gains energy, the mean angular velocity in the accelerator is decreased. The increase in velocity is more than compensated by the fact that the particle moves on a slightly longer path. For a linear accelerator $1/\gamma_t = 0$.

V. CYCLOTRON AND CYCLOTRON MASER

We consider now the classical cyclotron, in which the magnetic field is uniform. For such a field $\gamma_t = 1$, so that $m^* = -m_0/\beta^2$, a quantity which tends to infinity for nonrelativistic energies. The frequency Ω is zero, and the analysis presented so far breaks down. The rate of change of phase remains constant, and in the nonrelativistic case when the applied frequency equals the cyclotron frequency, the particle is continuously accelerated or decelerated in a spiral orbit. In the relativistic region, Ω is finite, and an ensemble of initially monoenergetic electrons in a uniform magnetic field (where there is no preferred equilibrium orbit) can absorb or give up energy by the mechanism described in section 3. Such a device is known as a cyclotron maser. Sprangle and Drobot[7] have given a detailed analysis. Their paper also contains an interesting historical discussion and a list of earlier references.)

It is interesting to compare the present approach with that of Borenstein and Lamb in their treatment of a classical laser.[8] They consider the orbiting particles as an ensemble of nonlinear oscillators, and in this way the connection with optical lasers is more direct. The present viewpoint, nevertheless, is perhaps more appropriate when making comparison with the free electron laser.

VI. FREE-ELECTRON LASER

In the devices studied so far, the condition $\dot{\phi}_0 = 0$ corresponds to equal particle and wave velocities, Eq. (3). This implies that the electromagnetic wave is a "slow" wave, with phase velocity less than c. This represents a severe limit at short wavelengths. Waves of this type necessarily have a *transverse* component of electric field, $E_\perp$, which increases rapidly from zero away from the symmetry axis. In order that $E_\perp$ should not exceed E_z, the transverse dimensions must be less than about $\lambda/3$.

This limit can be overcome by making use of a more complicated form of coherent interaction in which the wave moves faster than the particle; by this means wave types with $\omega/k_z = c$ or greater, which can be arbitrarily large in their transverse dimensions, are available. In order to provide this nonsynchronous, coherent interaction, it is necessary for either the particle orbit or the wave normal to depart from a straight line, and to be oscillatory in form. As a particular example we consider the configuration in the Stanford free-electron laser.[9,10] The electrons are constrained by means of a twisted transverse magnetic field to move in a helix of radius r and pitch λ_q, where $r \ll \lambda_q$. In the Stanford experiment these were 0.04 and 32 mm respectively. Consider now a circularly polarized plane wave traveling with wave normal along the axis of the helix. Relative to an electron, the wave (in the laboratory frame) moves forward with velocity $(1-\beta_z)c$, where β_z now represents the z-component of β. (For $r \ll \lambda_q$, $\beta_z \simeq \beta$; for highly relativistic electrons the difference may nevertheless be significant, as will be seen below.) The angular velocity of the polarization vector of the radiation at the electron is

$$\omega = \omega_r(1 - \beta_z) = ck_r(1 - \beta_z), \qquad (16)$$

where ω_r is the frequency of the radiation and k_r is its wavenumber; the angular velocity of the electron in the helical field is

$$\omega_q = \beta_z c k_q. \qquad (17)$$

When $\omega = \omega_q$ the interaction between the electron and wave is resonant and the rate of energy gain or loss depends on the phase of the electron. Since the electric field is transverse, the coupling with the particle is small, and the maximum rate of energy transfer is $\beta_z ceE_0(r/\lambdabar_q)$.

As an electron gains or loses energy from the field, the value of β_z increases or decreases, giving rise to phase motion in the z-direction of the type described by Eq. (5). Although the force that accelerates the electron is transverse, the magnetic field constrains the velocity change to be in the z-direction. (The action is as if the particle were in a smooth helical tube.) Since $r \ll \lambda_q$, the orbit and accelerating field are almost orthogonal and the interaction may be considered a second-order effect. This is the price to be paid for relaxing the constraint on the transverse dimensions.*

From Eqs. (16) and (17) the condition for synchronism is

$$\frac{\lambda_r}{\lambda_q} = \frac{1}{\beta_z} - 1. \qquad (18)$$

Using the relations

$$\beta_z = \beta\left(1 - \frac{r^2}{\lambdabar_q^2}\right)^{1/2}, \quad \gamma_x^2 = 1 - \beta_z^2, \qquad (19)$$

this may be written, when $\gamma \gg 1$,

$$\frac{\lambda_r}{\lambda_q} = \frac{1}{2\gamma^2}\left(1 + \frac{\gamma^2 r^2}{\lambdabar_q^2}\right) \qquad (20)$$
$$= \frac{1}{2\gamma^2}\left(1 + \frac{\lambda_q^2 e^2 B^2}{m_0^2 c^2}\right)$$

If $\gamma_\perp$ denotes the value of γ which the electron has when observed in a frame moving with velocity $\beta_z c$, then it may readily be shown that

$$\gamma = \gamma_z \gamma_\perp. \qquad (21)$$

If $\gamma r \ll \lambdabar$, the second term in the bracket of Eq. (20) is small, and $\gamma \simeq \gamma_z$, of $\gamma_\perp \simeq 1$.

In applying Eqs. (2), (3), (5) etc. to the free-electron laser, ω and k are given by

$$\omega = \omega_r, \quad k = k_q + k_r. \qquad (22)$$

The expression for k implies that the effective wavelength is less than that of the radiation. This can be seen as follows: if we take a snapshot of the system at fixed t, then we find that the orbit helix is in the opposite sense to that associated with the polarization vector, and the repeat length of the system is reduced by a factor $\lambda_q/(\lambda_r + \lambda_q)$ to the value $\lambda_q\lambda_r/(\lambda_q + \lambda_r)$.

The value of the effective mass of the particles can be found in a straightforward way by considering the kinematics of the system. We leave it to the reader to show that $m^* = \gamma\gamma_z^2 m_0$, which reduces (not un-

*A different constraint now operates. See Appendix 2.

expectedly) to the value $\gamma^3 m_0$ when the helix angle is such that $\gamma r/\lambda\!\!\!^{-}_q \ll 1$.

As a summary, values of m^*, Ω^2, and $\dot{\phi}_0/\Delta\gamma$ for some of the devices which have been discussed are shown in Table I. The mechanism used in the free-electron laser has also been proposed as an accelerator and is clearly described with diagrams by Palmer.[11] Nonsynchronous coherent interaction has also been employed in the Ubitron tube[11] and the cyclotron-wave tube.[12] It may be regarded as a parametric effect in which the field which produces the electron orbit modulation is a wave with zero frequency.[13]

TABLE I
Functional dependencies in wave devices.

Device	m^*	Ω^2	$\dot{\phi}_0/\Delta\gamma$
Nonrelativistic traveling-wave tube. Electrostatic plasma wave.	m_0	$\dfrac{eE_0k}{m_0}$	$\dfrac{\omega}{\beta^2}$
Cyclotron maser	$-\dfrac{\gamma m_0}{\beta^2}$	$\dfrac{2e^2EB\beta}{\gamma^2 m_0^2 c}$	$-\dfrac{\omega_c}{\gamma}$
Free-electron laser ($\gamma \gg 1$)	$\gamma\gamma_z^2 m_0$	$\dfrac{2e^2EB}{\beta_z\gamma^2 m_0^2 c}$	$\dfrac{\omega_r}{\beta_z^2\gamma_z^2\gamma}$

VII. LANDAU DAMPING

The wave–particle interaction described in sections 2 and 3 is responsible for Landau damping; if there is a distribution of particles $F(\dot{\phi}_0)$ initially uniform in ϕ, then energy is given to or extracted from the wave according to the sign of $(\partial F/\partial\dot{\phi}_0)$ at $\dot{\phi}_0 = 0$. The particular formalism of section 3 allows this to be found very simply. This general approach is a well-known alternative to Landau's original one.

First, we evaluate the integral of $<\dot{\phi} - \dot{\phi}_0>$ weighted with $F(\dot{\phi}_0)$ over $\dot{\phi}_0$. Since the main contribution comes from the region near $\dot{\phi}_0 = 0$, it is justifiable to make a Taylor expansion about this point. Normalizing $F(\dot{\phi}_0)$ such that its integral over $\dot{\phi}_0$ is unity, we have, making use of Eq. (10) with $x = \tfrac{1}{2}\dot{\phi}_0 t$,

$$\int_{-\infty}^{\infty}\left\{F(\dot{\phi}_0) + \dot{\phi}_0\left[\frac{\partial F}{\partial\dot{\phi}_0}\right]_{\dot{\phi}_0=0}\right\}<\dot{\phi} - \dot{\phi}_0> d\dot{\phi}_0$$

$$= \frac{\Omega^4 t^3}{4}\int_{-\infty}^{\infty}\frac{d}{dx}\left(\frac{\sin x}{x}\right)^2$$

$$\left(F + \frac{\partial F}{\partial\dot{\phi}}\frac{2x}{t}\right)\frac{2dx}{t} = \frac{\pi}{4}\left(\frac{\partial F}{\partial\dot{\phi}_0}\right)\Omega^4 t. \tag{23}$$

Making use of the nonrelativistic relation $\dot{\phi}_0 = \omega\Delta\gamma/\beta^2 = \omega\Delta\beta/\beta$ (in table) and substituting for Ω (in table) Eq. (23) may be written in terms of the energy loss per unit volume per unit time

$$\frac{dW}{dt} = \frac{n\beta^2 m_0 c^2}{\omega}\frac{\pi t}{4}\left(\frac{\partial F}{\partial\dot{\phi}_0}\right)\frac{e^2E_z^2k^2}{m_0^2}, \tag{24}$$

where n is the electron density. The attenuation constant α for the field amplitude is equal to $(1/2W)(dW/dt)$. Substituting for the stored energy per unit volume, (kinetic plus electrostatic), $W = \tfrac{1}{2}\varepsilon_0 E_z^2$ and for the plasma frequency, $\omega_p^2 = ne^2/\varepsilon_0 m$, we find that Eq. (24) becomes

$$\alpha = \frac{\pi}{2}\omega_p^2\omega\left[\frac{\partial F}{\partial\dot{\phi}_0}\right]_{\dot{\phi}=0}$$

$$= \frac{\pi}{2}\frac{\omega_p^2\omega}{k^2}\left[\frac{\partial f}{\partial v}\right]_{v=\omega/k}, \tag{25}$$

the result obtained by Landau.[15] It is clear from the limits on the validity of Eq. (10) that for large values of t, the damping coefficient α will vary with time. Such a situation has been analyzed by O'Neil[16] and Bailey and Denavit.[17]

VIII. CONCLUDING REMARKS

A general discussion of the interaction between a traveling electromagnetic wave and a distribution of noninteracting charged particles has been given. The relation between several devices which rely on such interaction has been explored, and a simple derivation given of the formula for Landau damping in a plasma.

It must be emphasized that the theory in this paper refers only to situations where the wave properties of the charged-particle medium can be ignored. In many practical devices, such as the traveling-wave tube operating in the small-signal regime, and free-electron lasers based on intense relativistic electron beams, this is not a valid assumption and a different type of analysis is required. A formal discussion of these two regimes may be found in papers by Kroll[18] and by Gover and Yariv.[19]

IX. ACKNOWLEDGEMENTS

Mr. J. Simkin and Mr. A. K. Stephens wrote the code and produced the curves shown in Fig. 2. Dr. R. N. Franklin and Mr. C. W. Planner made helpful comments on the subject matter.

APPENDIX 1

Calculation of $<\dot{\phi}(t) - \phi_0>$, Eq. (8)

The expression for the mean energy loss of particles uniformly distributed in ϕ_0 when $t = 0$ will be calculated for small values of t. Initially a series expansion of Eq. (8) will be developed. This equation may be written

$$\dot{\phi} - \dot{\phi}_0 = (\dot{\phi}_0^2 + 2\Omega^2(\cos\phi - \cos\phi_0))^{1/2} - \dot{\phi}_0 \qquad \text{(A1)}$$

When $2\Omega^2 \ll \dot{\phi}_0^2$ binomial expansion yields

$$\dot{\phi} - \dot{\phi}_0 = \frac{\Omega^2}{\dot{\phi}}(\cos\phi - \cos\phi_0) - \frac{\Omega^4}{2\dot{\phi}_0^3} \times (\cos\phi - \cos\phi_0)^2 + \ldots.. \qquad \text{(A2)}$$

This may be solved by iteration. The lowest order solution, $\phi = \phi_0 + \dot{\phi}_0 t$, is inserted in the first term on the right-hand side; the value of $\dot{\phi} - \phi_0$ is then integrated to give a more accurate value. This is then inserted into the first two terms to give a better value of $\phi - \phi_0$. Thus

$$\dot{\phi} - \dot{\phi}_0 = \frac{\Omega^2}{2\dot{\phi}_0}\{\cos(\phi_0 + \dot{\phi}_0 t) - \cos\phi_0\}. \qquad \text{(A3)}$$

Expanding and integrating, we find

$$\phi - \phi_0 = \frac{\Omega^2}{2\dot{\phi}_0}\left\{\cos\phi_0\left[\frac{\sin\dot{\phi}_0 t}{\dot{\phi}_0} - t\right] + \sin\phi_0\,\frac{\cos\dot{\phi}_0 t}{\dot{\phi}_0} - \frac{\sin\dot{\phi}_0}{\dot{\phi}_0}\right\}, \qquad \text{(A4)}$$

This value of $\phi - \phi_0$ may now be inserted into Eq. (A2), retaining the first two terms on the right-hand side to give the next approximation. This gives rise to a lengthy expression, which contains terms having factors of the form

$$\sin\left\{\left(\frac{\Omega^2}{2\dot{\phi}_0^2}\right)\sin(\dot{\phi}_0 t)\right\}$$

(and others with cos in place of either or both sines). The condition

$$(\dot{\phi}_0 t)\left(\frac{\Omega^2}{2\dot{\phi}_0^2}\right) \ll 1 \qquad \text{(A5)}$$

now imposes a condition for the validity of the expansion at this stage. Integration over ϕ_0 removes terms of the order t^2; fourth-order terms, however, remain to give Eq. (10). This analysis may be compared with those of Colson,[10] and Yariv and Shih.[20]

APPENDIX 2

Constraint on transverse dimensions in free electron laser

For practical purposes it is important to know the nature of the constraint on the transverse dimensions in the free electron laser. For illustration, we consider helical geometry for the external magnetic field. On the axis, the magnetic field has components $(B_0\cos(z/\lambda\!\!\!^-_q),\ B_0\sin(z/\lambda\!\!\!^-_q),\ 0)$. The conditions that div B and curl B should be zero imply that the mean field increases with a, the distance from the axis; for small a, $\delta B = \frac{1}{2}B_0 a^2/\lambda\!\!\!^-_q{}^2$. This implies that β_z is a function of a. Although the variation is small, a phase-slip of more than half an *optical* wavelength over the interaction region is intolerable. Making use of Eqs. (19) and (20), and the fact that $B \propto \gamma r/ƛ_q$, it is not difficult to show that

$$\Delta\beta_z = r^2 a^2/ƛ_q^4 \tag{B1}$$

where a and r, ($\ll a$), are the distance from the axis and the radius of the orbit helix respectively.

This quantity must be less than $\frac{1}{2}(1 - \beta_z)\lambda_r/L$ if the phase-slip is to be less than $\lambda_r/2$ in a device of length L. Using Eq. (18) this may be written

$$a^2 < \frac{\pi}{2}\left(1 + \frac{1}{K^2}\right)\frac{ƛ_q^3}{L} \quad . \tag{B3}$$

REFERENCES

1. C.W. Planner, *Phys. Letts.* **67A**, 263 (1978).
2. S.E. Webber, *IRE Trans. Elec. Devices* **ED-4**, 87 (1957).
3. J.D. Lawson, *The Physics of Charged-Particle Beams* (Clarendon Press, Oxford, 1977).
4. R.B. Neal, *The Stanford Two-mile Accelerator* (W.A. Benjamin, New York, 1968).
5. R.H. Helm and G.A. Leow, *Beam Breakup* in *Linear Accelerators,* P.M. Lapostolle and A.L. Septier, Eds. (North Holland, Amsterdam 1970, p. 173).
6. P.B. Wilson, *A Study of Beam Blow-up in Electron Linacs*, Stanford memorandum HEPL-297, 1963.
7. P. Sprangle and A.T. Drobot, *IEEE Trans. Microwave Theory and Tech.* **MTT 25,** 528 (1977).
8. M. Borenstein and W.E. Lamb, *Phys. Rev.* **A3**, 1298 (1972).
9. D.A.G. Deacon, L.R. Elias, J.M.J. Madey, G.J. Ramain, H.A. Schwettman, and T.I. Smith, *Phys. Rev.Lett.* **38,** 892 (1977).
10. W.B. Colson, *Phys. Letts.* **64A**, 190 (1977) and *Physics of Quantum Electronics* **5**, 115 (Addison-Wesley, Reading, Mass., 1977).
11. R.B. Palmer, *J. Appl. Phys.* **43**, 3014 (1972).
12. R.M. Phillips, *IRE Trans. Elec. Devices* **ED-7**, 231 (1960).
13. W.H. Louisell, *Coupled Mode and Parametric Electronics,* (John Wiley & Sons, New York, 1960).
14. J. Dawson, *Phys. Fluids* **4**, 869 (1961).
15. L. Landau, *J. Phys. (USSR)* **10,** 15, (1946).
16. T. O'Neil, *Phys. Fluids* **8**, 2255 (1965).
17. V.L. Bailey, Jr. and J. Denavit, *Phys. Fluids* **13**, 451 (1970).
18. N.F. Kroll, *Physics Quantum Electronics* **5**, 115 (Addison-Wesley, Reading, Mass., 1978).
19. A. Gover and A. Yariv, as Ref. 18, p. 197.
20. A. Yariv and C.C. Shih, *Optics Communications* **24**, 233 (1978).